On the Convexity and Feasibility of the Bounded Distortion Harmonic Mapping Problem

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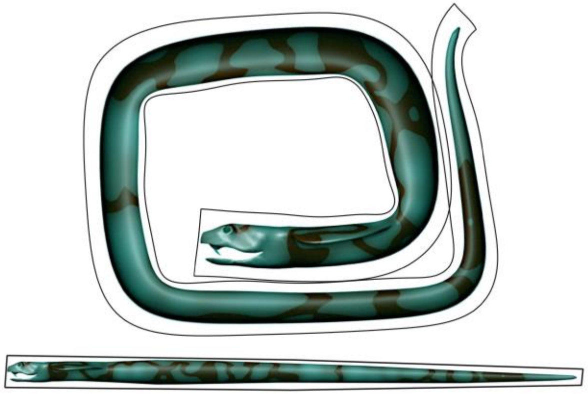


Motivation

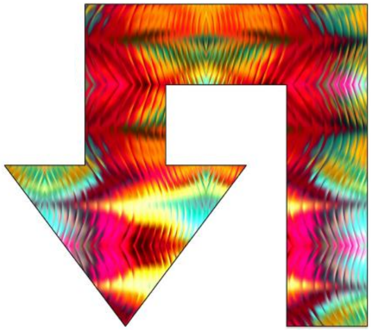
Requirements from a high quality mapping

- Smooth
- Locally injective
- Bounded distortion

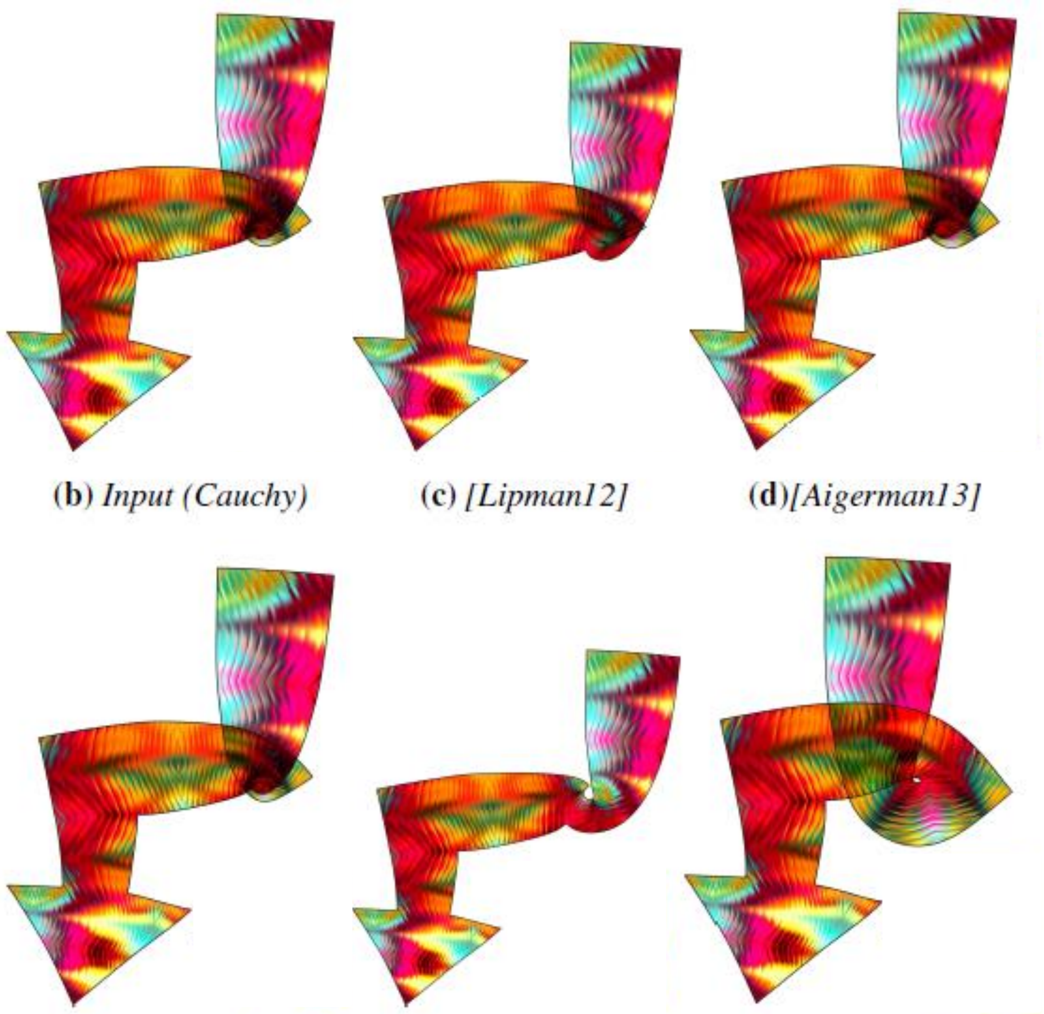
The quality is important for applications such as

- Producing pleasing results (e.g. shape deformation or texture mapping)
- Physical simulation
- Remeshing

The projection problem

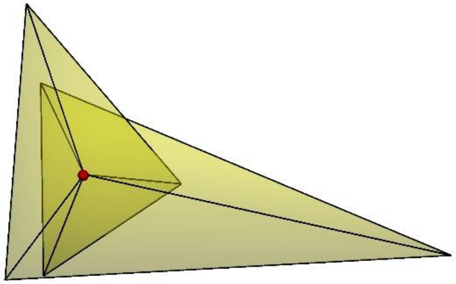


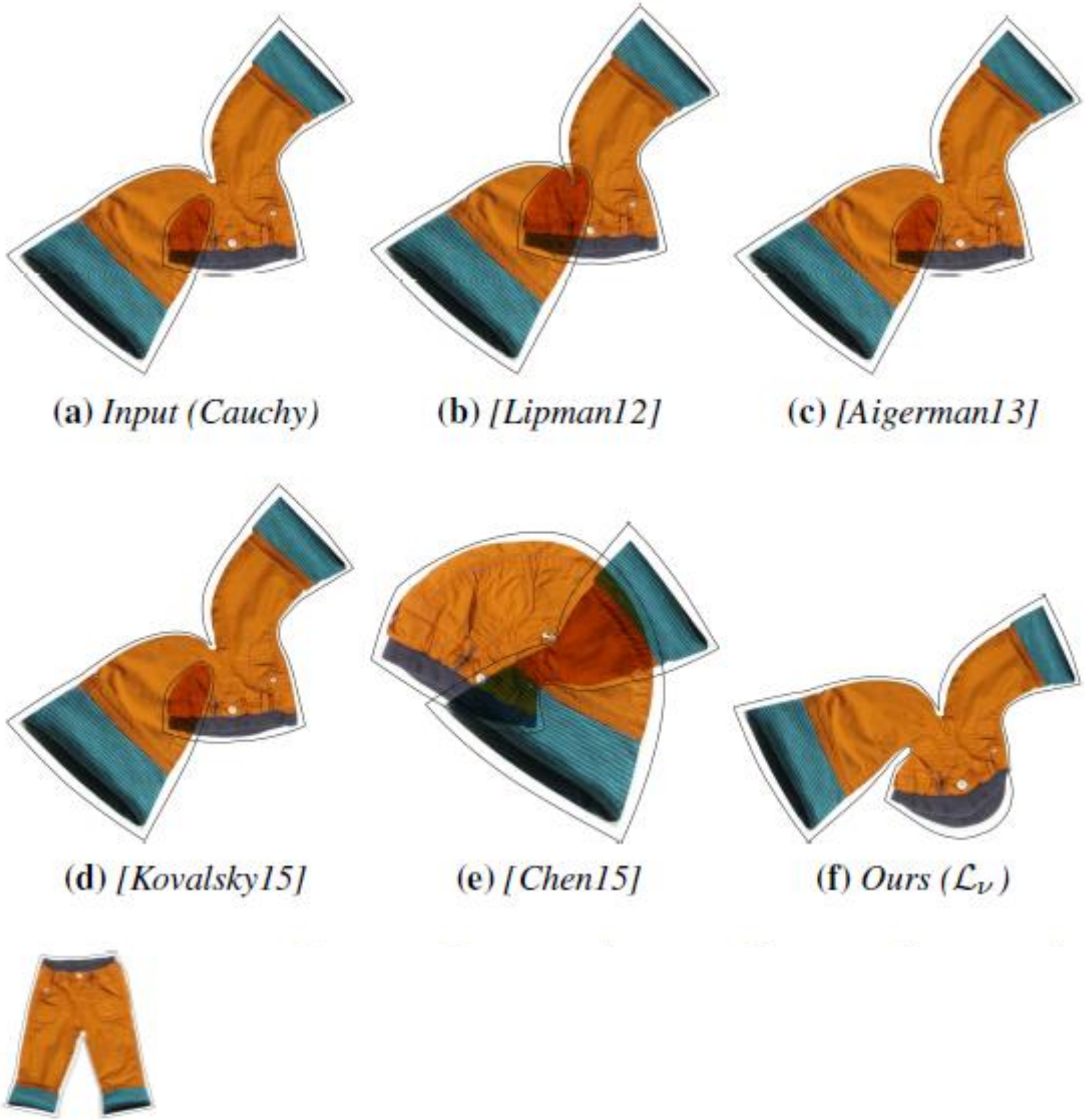
Domain



(b) *Input (Cauchy)* (c) *[Lipman12]* (d)*[Aigerman13]*

(e) *[Kovalsky15]* (f) *[Chen15]* (g) *Ours* ($\mathcal{L}_\nu$)





(a) *Input (Cauchy)* **(b)** *[Lipman12]* **(c)** *[Aigerman13]*

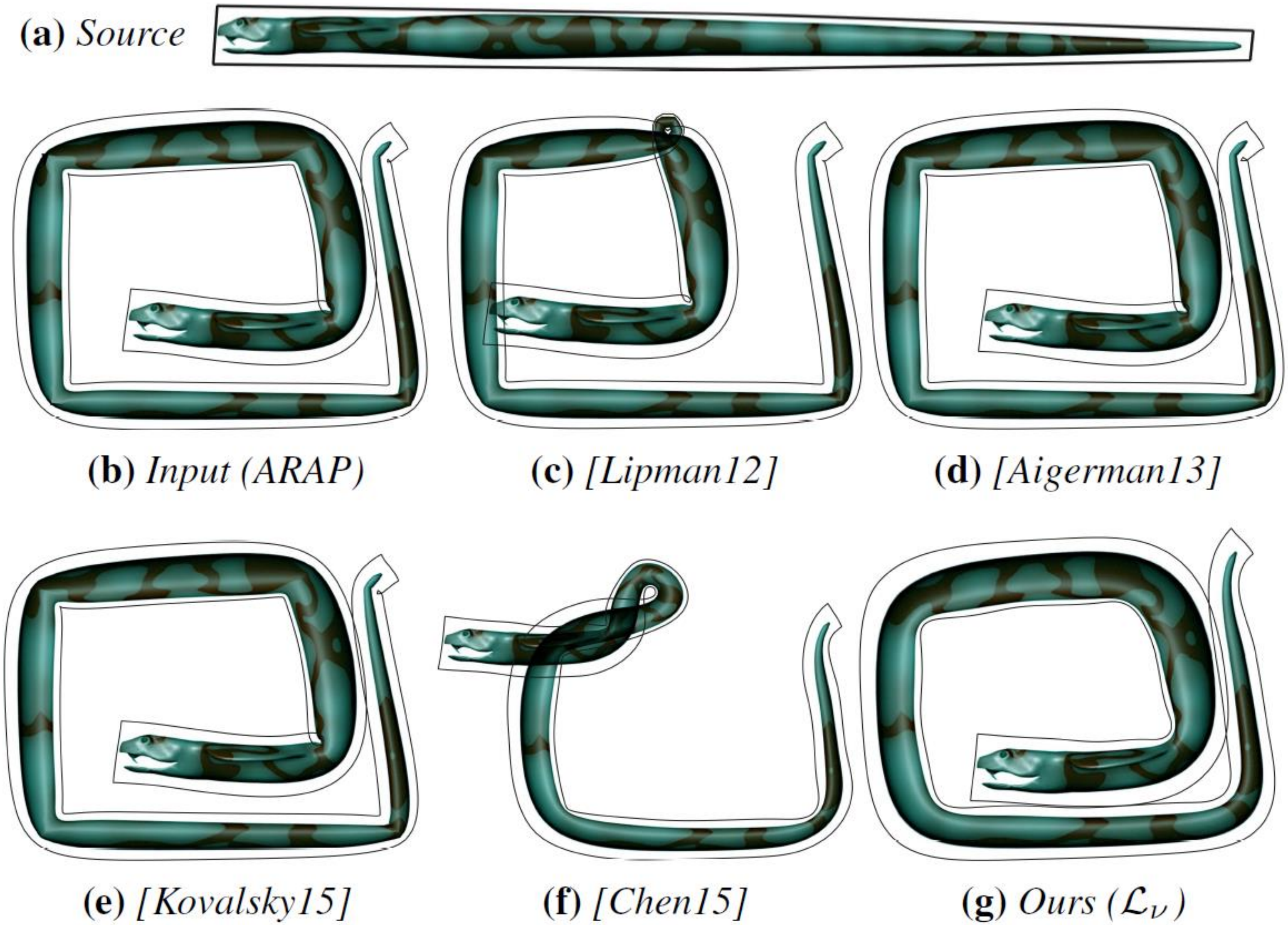
(d) *[Kovalsky15]* **(e)** *[Chen15]* **(f)** *Ours ($\mathcal{L}_\nu$)*

(g) *Domain*

[Lipman12] iterations

Initialization

(a) *Iter 1* **(b)** *Iter 2* **(c)** *Iter 3* **(d)** *Iter 4* **(e)** *Iter 5*

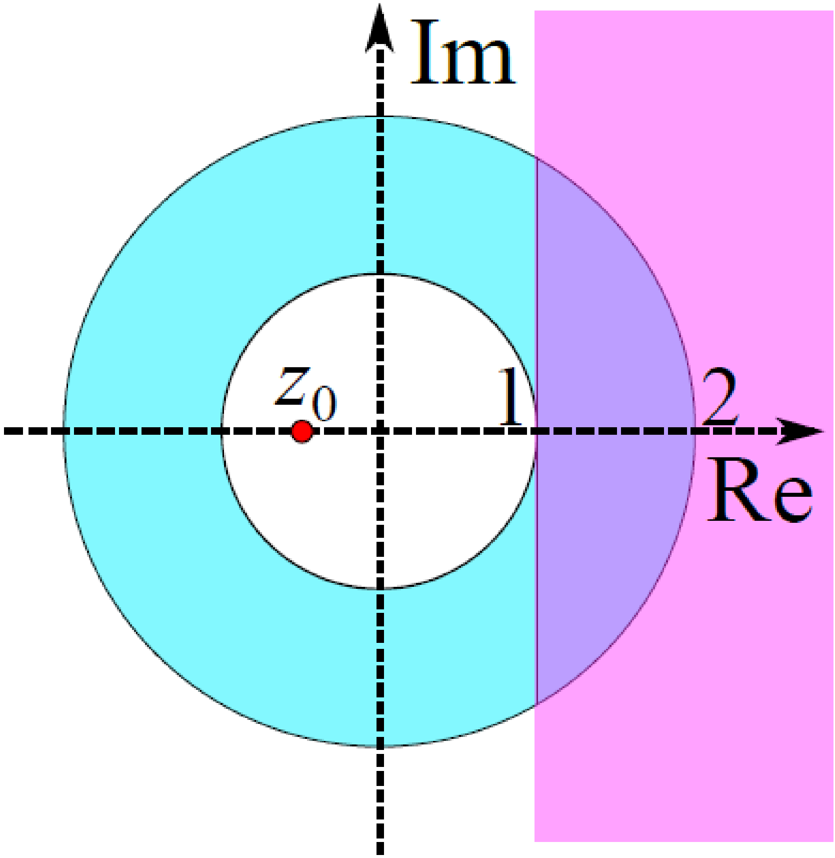

(a) Source
(b) Input (ARAP)
(c) [Lipman12]
(d) [Aigerman13]
(e) [Kovalsky15]
(f) [Chen15]
(g) Ours ($\mathcal{L}_\nu$)

A convex problem

- Can be solved efficiently
- No initial feasible point is needed
- Achieving a global minimum is guaranteed (if it exists)

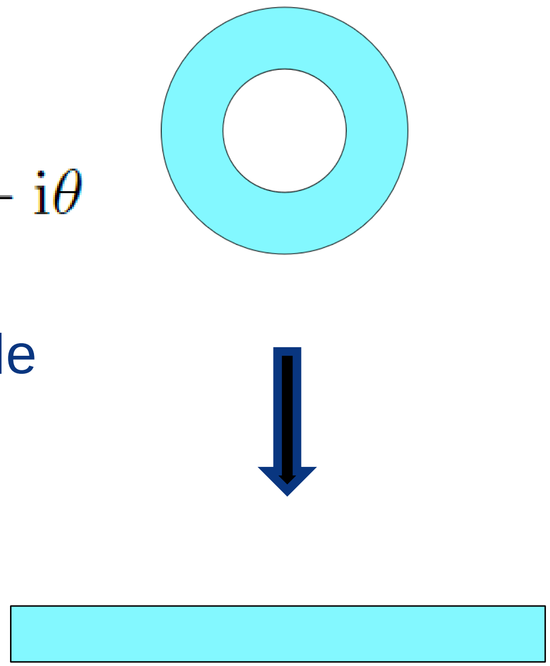
A mock example

$$\min_{z \in \mathbb{C}} \quad |z - z_0|$$
$$s.t. \quad 1 \leq |z|$$
$$|z| \leq 2$$



Principal branch of the complex logarithm operator

$$\mathrm{Log}(z) = \ln|z| + \mathrm{i}\mathrm{Arg}(z) = l + \mathrm{i}\theta$$



Maps the annulus bijectively to the convex rectangle

$$[0, \ln(2)] \times (-\pi, \pi]$$

Now the objective function is not convex

$$|\mathrm{e}^{l+\mathrm{i}\theta} - z_0|$$

Alternative convex problem

$$\begin{aligned} \min_{l,\theta} \quad & |l + \mathrm{i}\theta - \mathrm{Log}(-0.5)| \\ s.t. \quad & 0 \le l \le \ln(2) \\ & -\pi < \theta \le \pi. \end{aligned}$$

Bounded distortion harmonic mapping ($\mathcal{BD}$ space)

$$f : \Omega \subset \mathbb{C} \to \mathbb{C}$$

$$k_f(z) = \frac{|f_{\bar{z}}(z)|}{|f_z(z)|} \leq k < 1 \qquad \forall z \in \Omega$$

$$\Sigma_f(z) = |f_z(z)| + |f_{\bar{z}}(z)| \leq \Sigma < \infty \qquad \forall z \in \Omega$$

$$0 < \sigma \leq |f_z(z)| - |f_{\bar{z}}(z)| = \sigma_f(z) \qquad \forall z \in \Omega$$

k, Σ, σ *are real constants*

The space is not convex

$\mathcal{H}$ space

A pair of a holomorphic and a real functions

$$h = \{\Psi(z), r(w)\}$$

is in $\mathcal{H}$ if

$$\left|\Psi'(w)\right| \leq k\, r(w) \qquad \forall w \in \partial\Omega$$

$$\left|\Psi'(w)\right| \leq \Sigma - r(w) \qquad \forall w \in \partial\Omega$$

$$\left|\Psi'(w)\right| \leq r(w) - \sigma \qquad \forall w \in \partial\Omega$$

$\mathcal{H}$ is convex

Mapping between the two spaces

Operator definition $\mathcal{F} : \mathcal{BD} \to \mathcal{H}$

- Harmonic mapping decomposition to holomorphic and anti-holomorphic

$$f(z) = \Phi(z) + \overline{\Psi}(z)$$

- Map

$$\{\Psi(z), r(w)\} \; = \; \{\Psi(z), |\Phi'(w)|\}$$

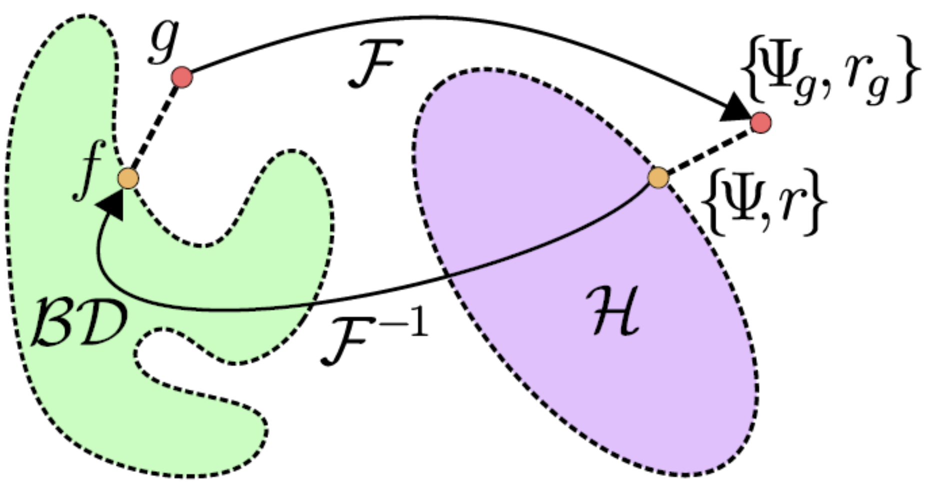
The operator is a bijection

Optimization

Given bounds and a mapping g, project it onto $\mathcal{BD}$

Energy

$$\oint_{\partial\Omega} \Big(r(w) - |g_z(w)| \Big)^2 ds + \lambda_{\mathcal{H}} \iint_{\Omega} \Big| \Psi(z)' - \overline{g_{\bar{z}}}(z) \Big|^2 da$$

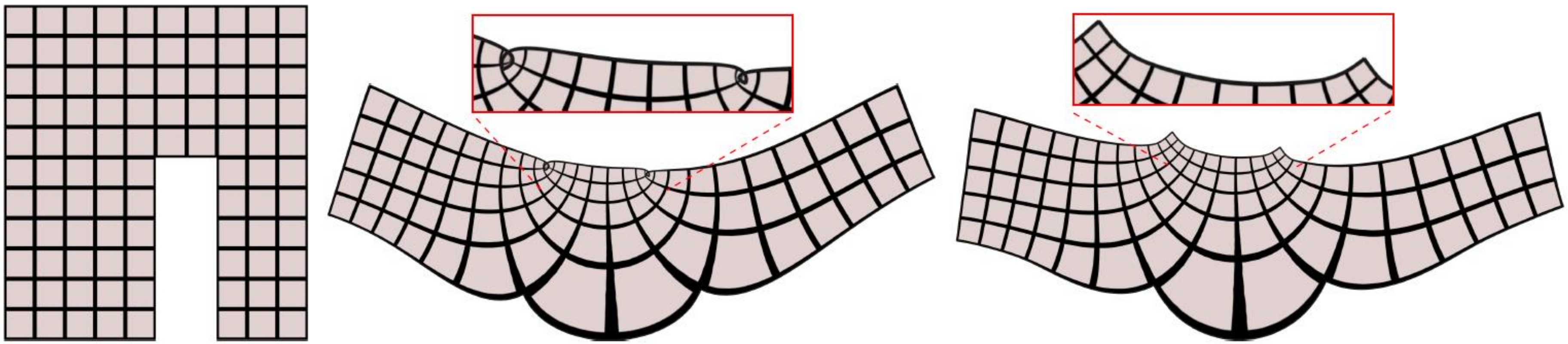


BEM discretization (based on [Weber09])

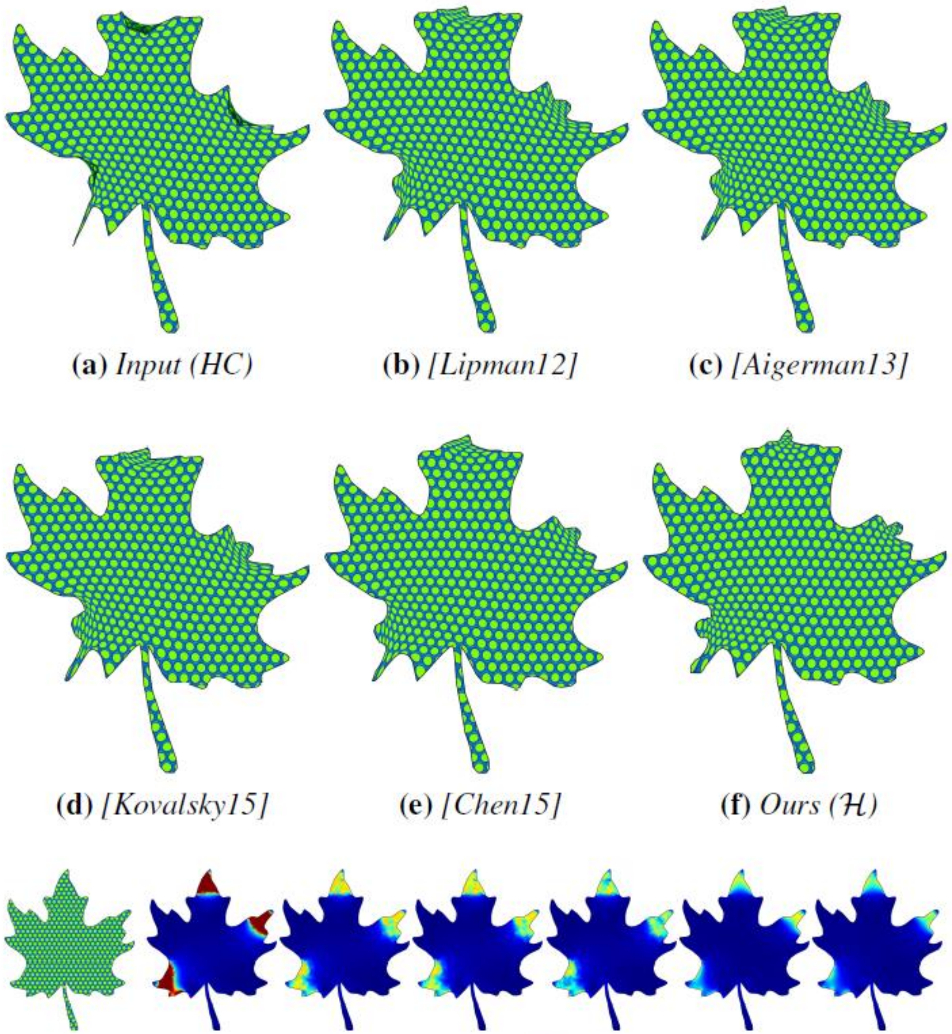
$$\min_{\psi_1..\psi_n, r_1..r_{|\mathcal{A}|}} \quad \mathrm{E}_{\mathcal{H}}$$

$$\begin{aligned} s.t. \quad & \Psi(z_0) = 0, \\ & \forall p_i \in \mathcal{A} \quad \left|\Psi'(p_i)\right| \leq k\, r_i, \\ & \forall p_i \in \mathcal{A} \quad \left|\Psi'(p_i)\right| \leq \Sigma - r_i \\ & \forall p_i \in \mathcal{A} \quad \left|\Psi'(p_i)\right| \leq r_i - \sigma \end{aligned}$$

$$\mathrm{E}_{\mathcal{H}} = \sum_{i=1}^{|\mathcal{A}|} \Big(r_i - |g_z(p_i)| \Big)^2 + \lambda_{\mathcal{H}} \sum_{i=1}^{|\mathcal{B}|} \Big| \Psi'(p_i) - \overline{g_{\bar{z}}}(p_i) \Big|^2$$



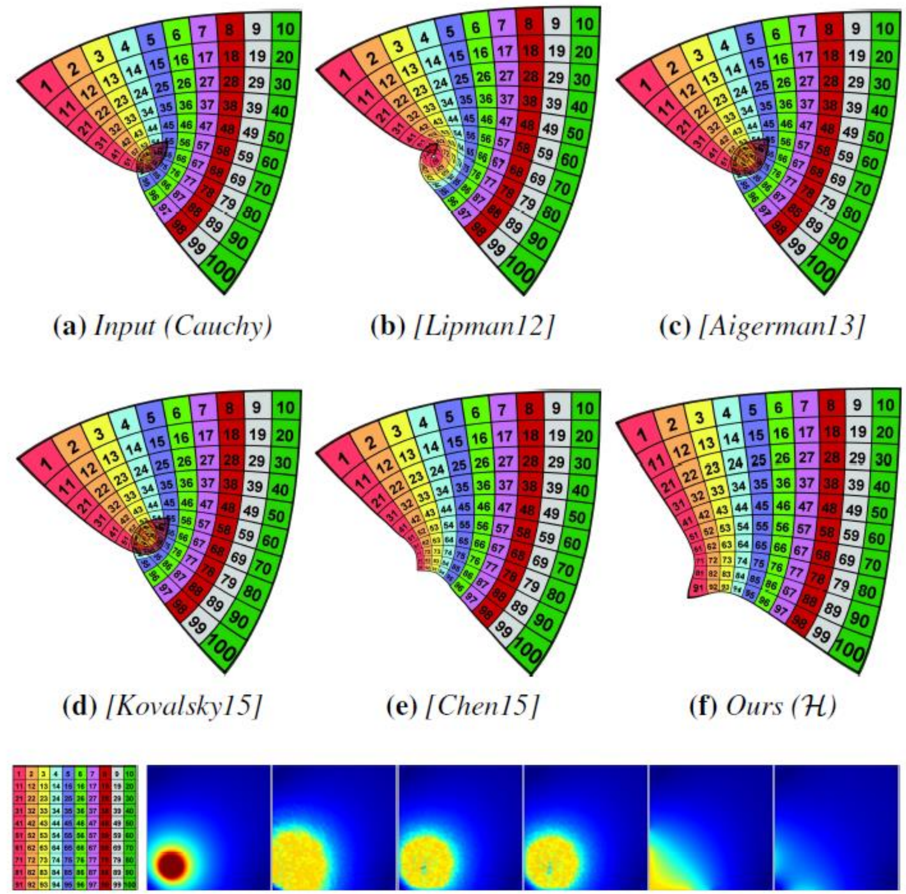
(a) Domain (b) Input (Cauchy) (c) Ours $(\mathcal{H})$



(a) *Input (HC)* **(b)** *[Lipman12]* **(c)** *[Aigerman13]*

(d) *[Kovalsky15]* **(e)** *[Chen15]* **(f)** *Ours ($\mathcal{H}$)*

(g) **(h)**



(a) *Input (Cauchy)* **(b)** *[Lipman12]* **(c)** *[Aigerman13]*

(d) *[Kovalsky15]* **(e)** *[Chen15]* **(f)** *Ours ($\mathcal{H}$)*

(g) **(h)**

A drawback

The energy is oblivious to the argument of g_z

$$\mathrm{E}_{\mathcal{H}} = \sum_{i=1}^{|\mathcal{A}|} \Big(r_i - |g_z(p_i)| \Big)^2 + \lambda_{\mathcal{H}} \sum_{i=1}^{|\mathcal{B}|} \Big| \Psi'(p_i) - \overline{g_{\bar{z}}}(p_i) \Big|^2$$

The logarithmic $\mathcal{L}_\nu$ space

Based on two holomorphic functions

$$l(z) = \log f_z$$

$$\nu(z) = \frac{\overline{f_{\bar{z}}}}{f_z} = \frac{\overline{f_{\bar{z}}}}{\mathrm{e}^l}$$

Inequalities satisfied at every boundary point

$$k_f(w) = |\nu(w)| \leq k \qquad \forall w \in \partial\Omega$$

$$\Sigma_f(w) = \mathrm{e}^{\mathrm{Re}(l(w))}(1 + |\nu(w)|) \leq \Sigma \qquad \forall w \in \partial\Omega$$

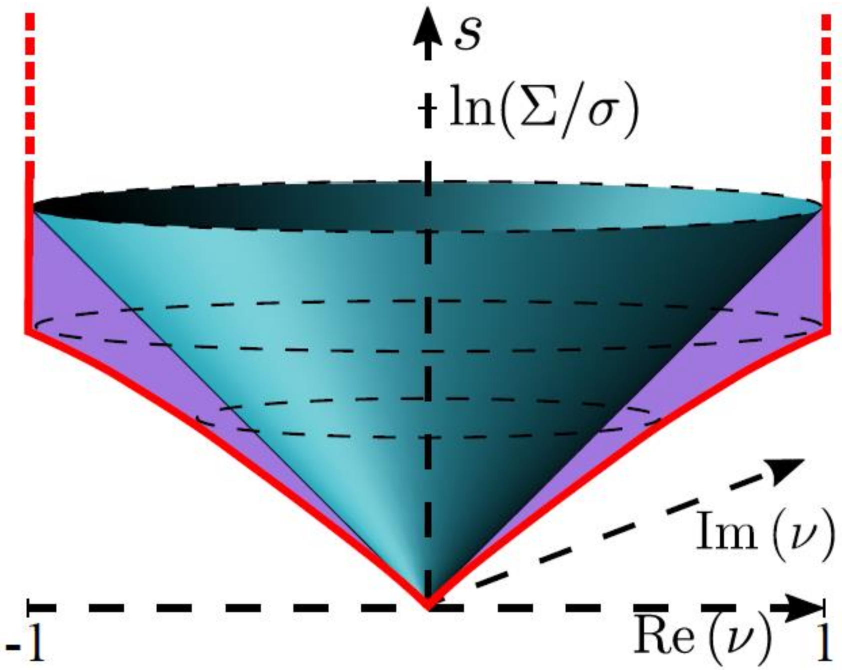
$$\sigma \leq \mathrm{e}^{\mathrm{Re}(l(w))}(1 - |\nu(w)|) = \sigma_f(w) \qquad \forall w \in \partial\Omega$$

Has one-to-one correspondence with $\mathcal{BD}$

Convexity

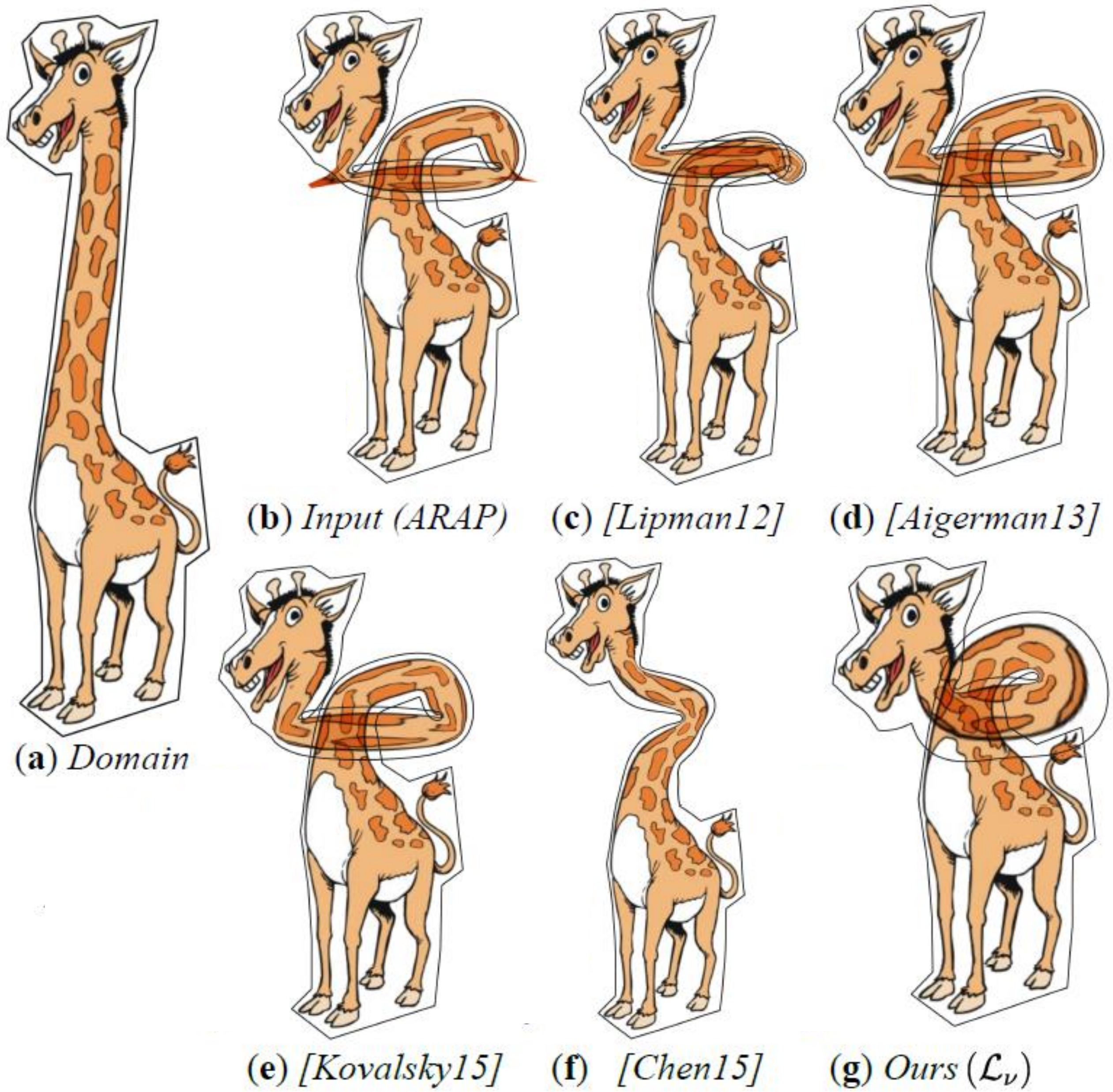
The second constraint is not convex

Substitute it with a second order cone



Advantages over the convexification of [Lipman12]

- Does not depend on local frames
- [Lipman12] convexifies the constraints for both k_f and σ_f while we only convexify the constraint for Σ_f
- Our space is nonempty – the optimization problem is always feasible while [Lipman12] requires a feasible starting point.


(a) Domain
(b) Input (ARAP)
(c) [Lipman12]
(d) [Aigerman13]
(e) [Kovalsky15]
(f) [Chen15]
(g) Ours ($\mathcal{L}_\nu$)

The logarithmic $\mathcal{L}_\psi$ space

Based on two holomorphic functions $\{l(z), \Psi(z)\}$

Inequalities satisfied at every boundary point

$$\left|\Psi'(w)\right| \leq k\, \mathrm{e}^{\mathrm{Re}(l(w))} \qquad \forall w \in \partial\Omega$$

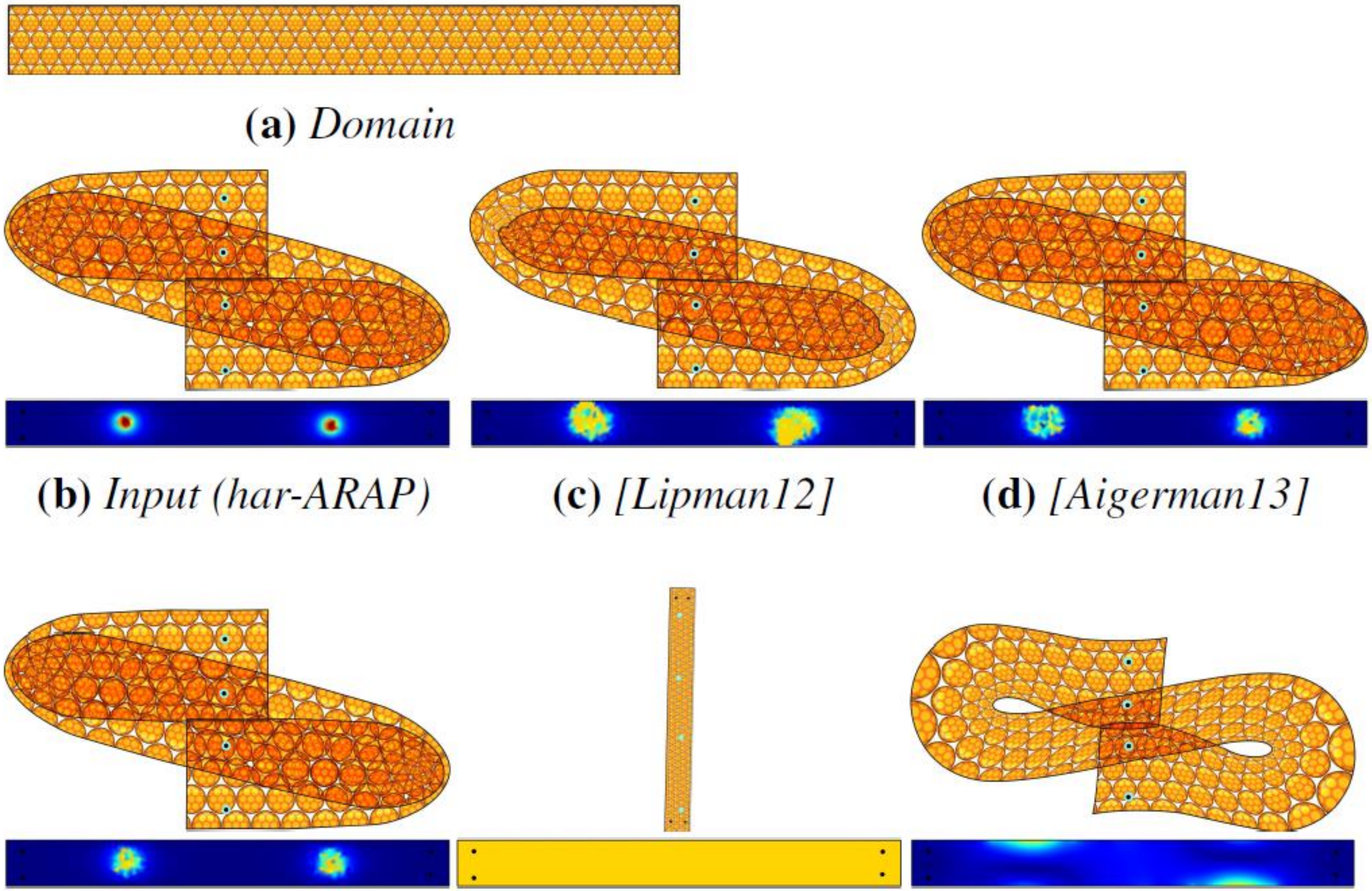
$$\mathrm{e}^{\mathrm{Re}(l(w))} + \left|\Psi'(w)\right| \leq \Sigma \qquad \forall w \in \partial\Omega$$

$$\sigma \leq \mathrm{e}^{\mathrm{Re}(l(w))} - \left|\Psi'(w)\right| \qquad \forall w \in \partial\Omega$$

Not convex

Solve a convex problem to find $\Phi(z)$, then solve a second convex problem to find $\Psi(z)$

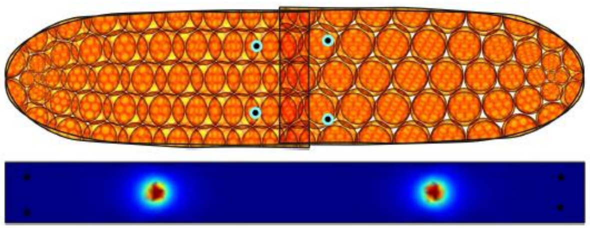
Supports p2p-constraints



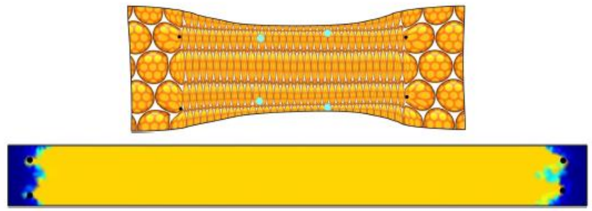
(a) *Domain*

(b) *Input (har-ARAP)*　**(c)** *[Lipman12]*　**(d)** *[Aigerman13]*

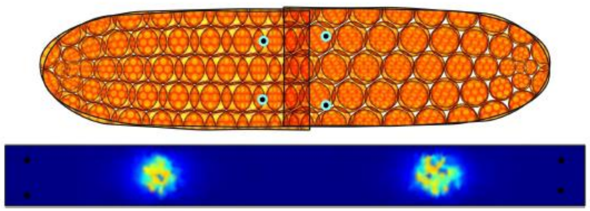
(e) *[Kovalsky15]*　**(f)** *[Chen15]*　**(g)** *Ours* $(\mathcal{L}_\psi)$



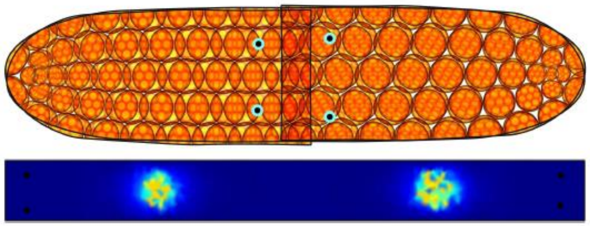
(a) *Input (har-ARAP)*



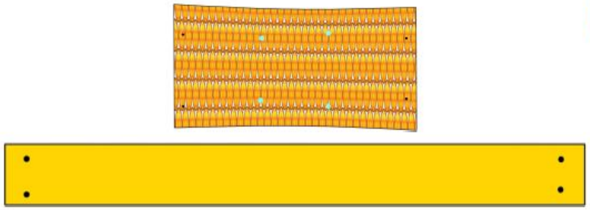
(b) *[Lipman12]*



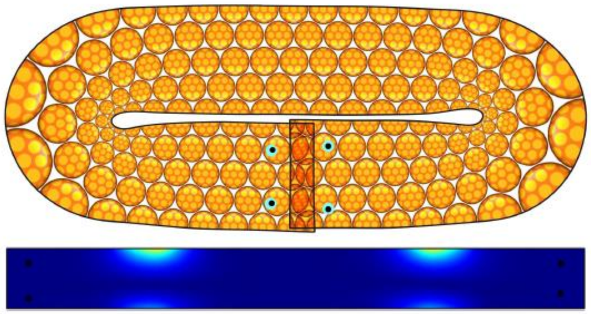
(c) *[Aigerman13]*



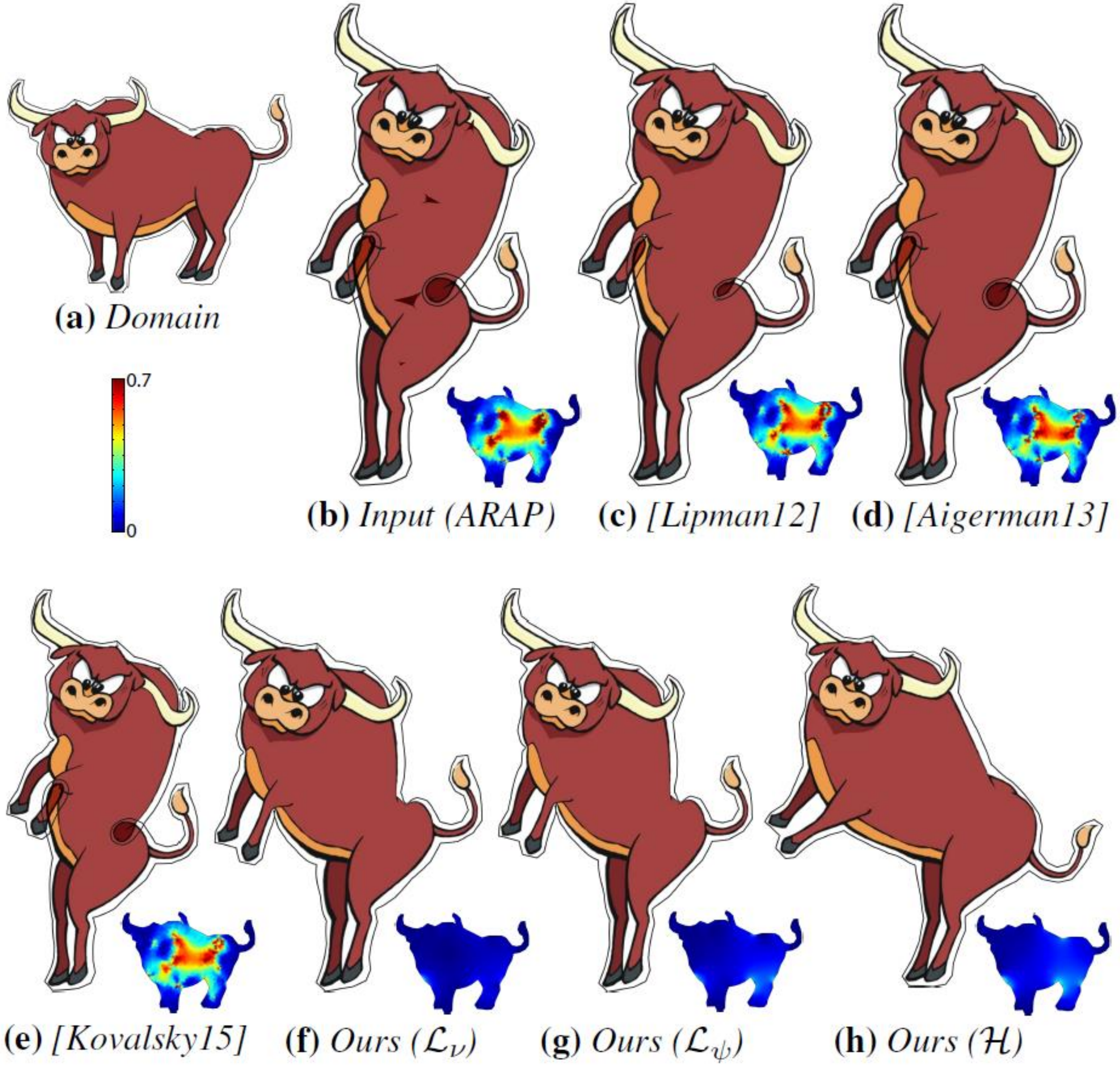
(d) *[Kovalsky15]*



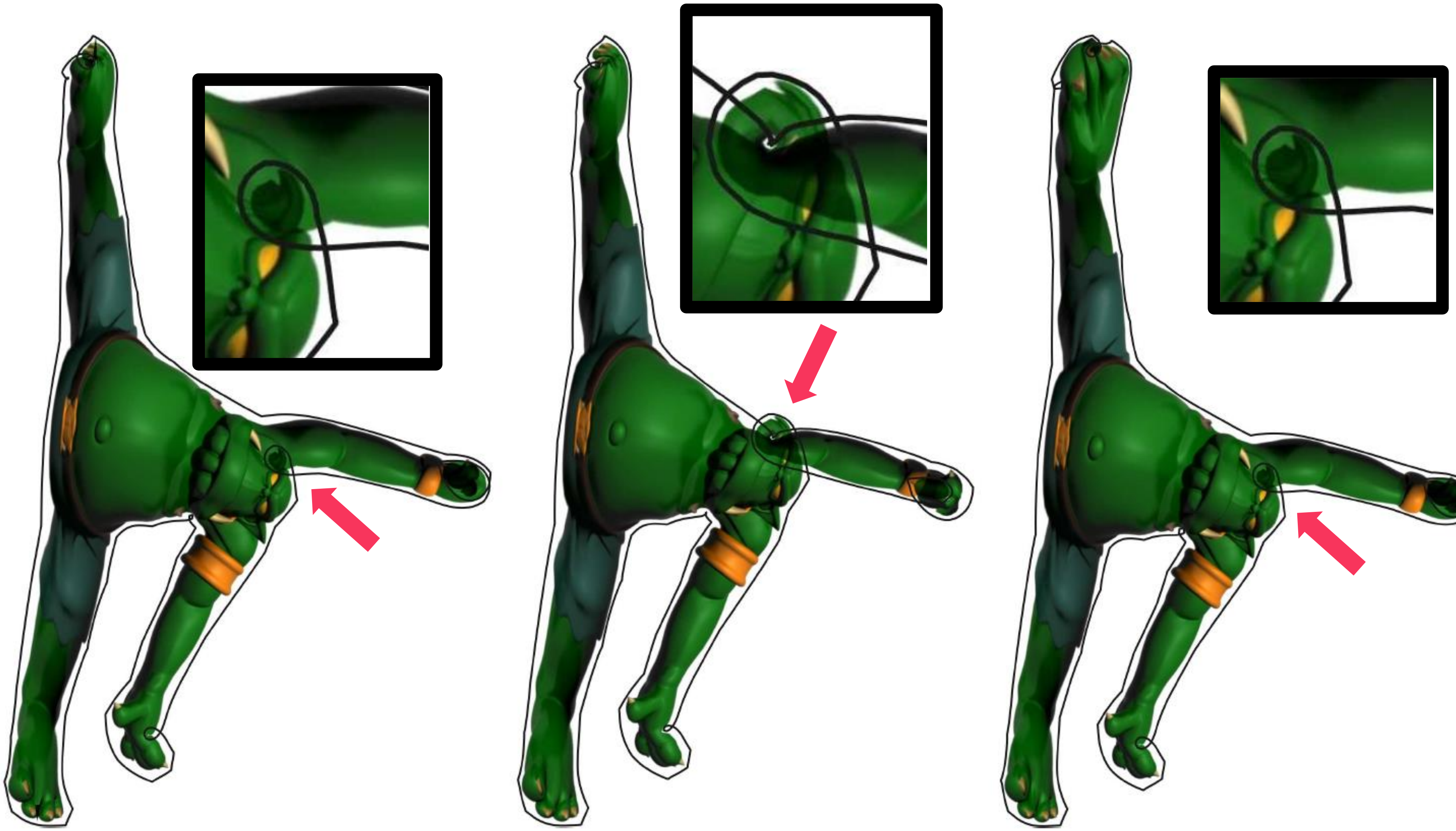
(e) *[Chen15]*



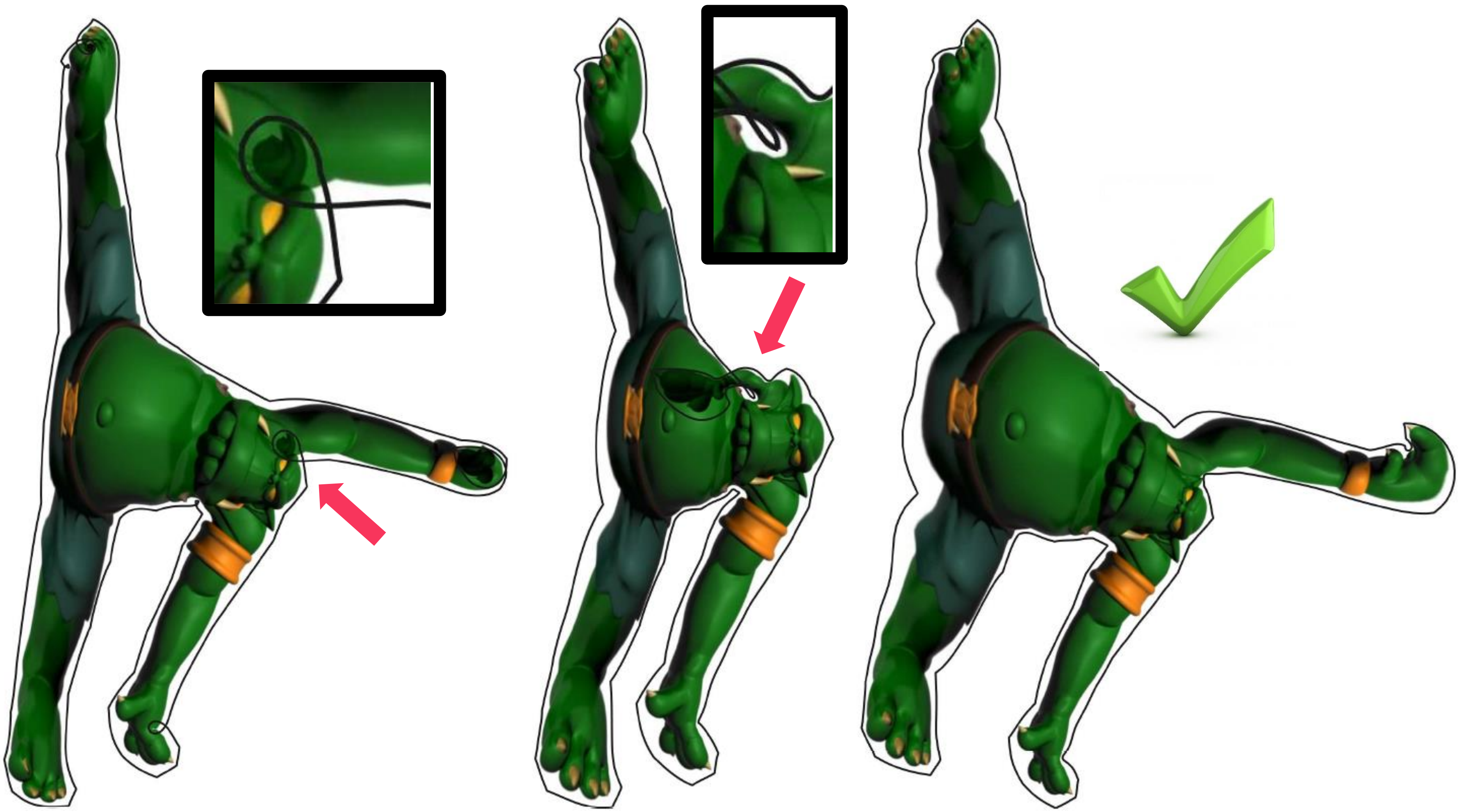
(f) *Ours* ($\mathcal{L}_\psi$)



(a) Domain
0.7
0
(b) Input (ARAP)
(c) [Lipman12]
(d) [Aigerman13]
(e) [Kovalsky15]
(f) Ours ($\mathcal{L}_\nu$)
(g) Ours ($\mathcal{L}_\psi$)
(h) Ours ($\mathcal{H}$)



(a) *Input (ARAP)* (b) *[Lipman12]* (c) *[Aigerman13]*



(d) *[Kovalsky15]* **(e)** *[Chen15]* **(f)** *Ours* ($\mathcal{L}_\nu$)

Conclusion

- Summary
  - Characterizing the nonconvex space of bounded distortion harmonic mappings using three spaces
- Future work
  - Find alternative spaces to similar problems