ARAP-L_∞ Dual SOCP Formulation

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Primal P^* formulation:

$$\begin{aligned} &\min_{t,x} t \\ s.t. \quad & Ax = b \\ & ||d_i|| \leq t, \quad d_i(x) = G_i x - r_i \end{aligned}$$

where x is a column stack of UV coordinates; G_i is the mesh gradient matrix w.r.t. to facet i; r_i is the column stack of the 2×2 matrix R_i that holds the target frame; $i = 1..m$; m number of facets.

Due to sparsity pattern it is more efficient to solve the dual problem.

P^* Lagrangian:

$$\mathcal{L} = c^T \begin{bmatrix} t \\ x \end{bmatrix} + \mu^T (Ax - b) + \sum_{i=1}^{m} \lambda_i (||d_i|| - t)$$

where $c^T = [10...0], \quad c^T \begin{bmatrix} t \\ x \end{bmatrix} = t.$

The primal problem:

$$P^* = \min_{x} \max_{\lambda \geq 0;\, \mu} \mathcal{L}$$

since μ is under max, it would penalize and make the term go to infinity, whenever $Ax - b$ is not 0. λ, on the other hand, we allow to be only positive so it could penalize the constraint when it's positive. For example, that's how the barrier method works by composing the constraint with a function that goes to zero when the constraint is negative, and to infinity otherwise.

We observe that

$$\max_{\lambda_i > 0} \lambda_i (||d_i|| - t) = \max_{||u_i|| < \lambda_i} u_i^T d_i - t_i \lambda_i$$

due to

$$u_i^* = \frac{d_i}{||d_i||} \lambda_i \Longrightarrow (u_i^*)^T d_i - t\lambda = ||d_i|| - t\lambda_i$$

and we get

$$P^* = \min_x \max_{\lambda \geq ||u_i||;\, \mu} c^T \begin{bmatrix} t \\ x \end{bmatrix} + \mu^T(Ax - b) + \sum_{i=1}^{m} (u_i^T d_i - t_i \lambda_i)$$

Now we derive the dual problem. From weak duality $P^* \geq D^*$(strong duality they are equal).

$$D^* = \max_{\lambda \geq ||u_i||;\, \mu} \min_x \mathcal{L}$$

The inner problem is an unconstrained minimization in x. We find optimality conditions.

$$\frac{\partial \mathcal{L}}{\partial x} = c + [0A]^T \mu + \sum_i ([0G_i]^T u_i - \lambda_i c)$$

Thus the optimal solution for the inner problem satisfies $\frac{\partial \mathcal{L}}{\partial x} = 0$. Since the objective is a linear function in x, since the derivative is zero, it means that the coefficients of x in the objective sum to zero, and x is eliminated from the objective. If the objective was not linear, we would have found an expression for x and assign it in the objective.

We get the dual problem:

$$\begin{gathered} \max_{\lambda, u_i} -\mu^T b - \sum_i u_i^T r_i \\ s.t. \;\; \frac{\partial \mathcal{L}}{\partial x} = 0 \\ ||u_i|| < \lambda_i \end{gathered}$$

The dual Lagrangian:

$$\mathcal{L}^* = -\mu^T b - \sum_i u_i^T r_i + \begin{bmatrix} t \\ x \end{bmatrix} \frac{\partial \mathcal{L}}{\partial x} + \sum_i \phi_i(||u_i|| - \lambda_i)$$

We can't express x explicitly, but we can retrieve it from the primal-dual interior point optimizer.